Algebraic Relations and Integrality of Limit Sets of Maximal Cusp Groups

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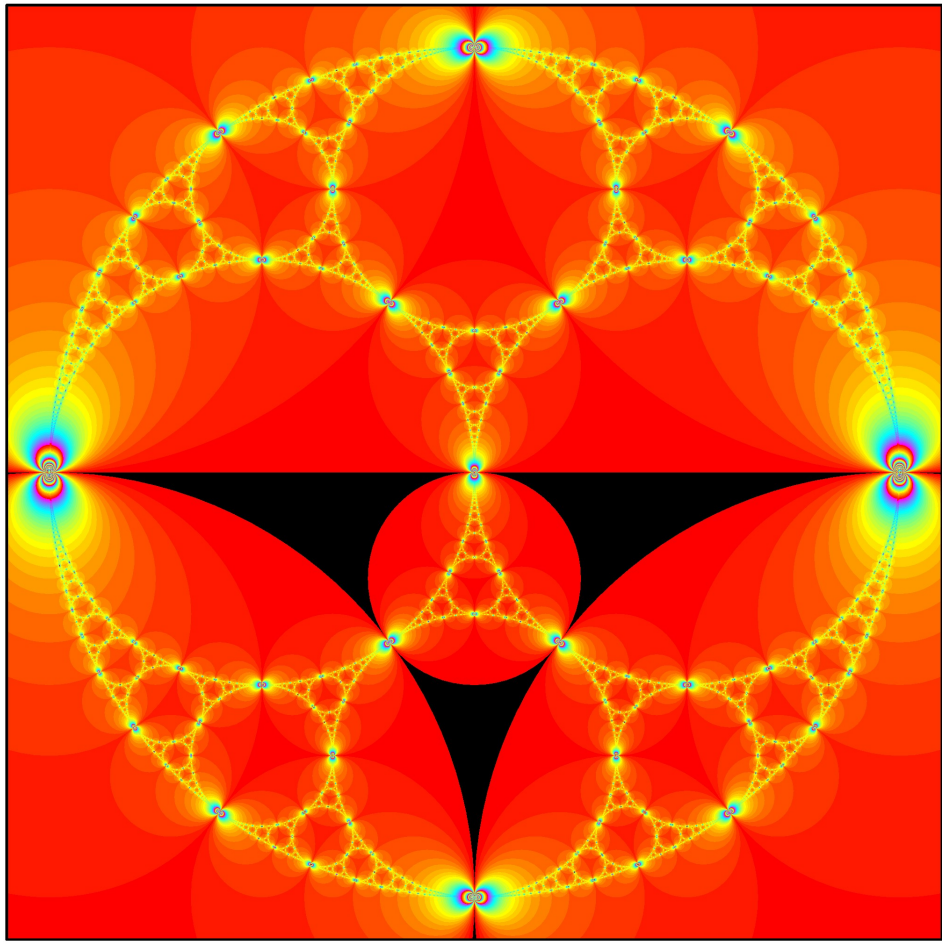
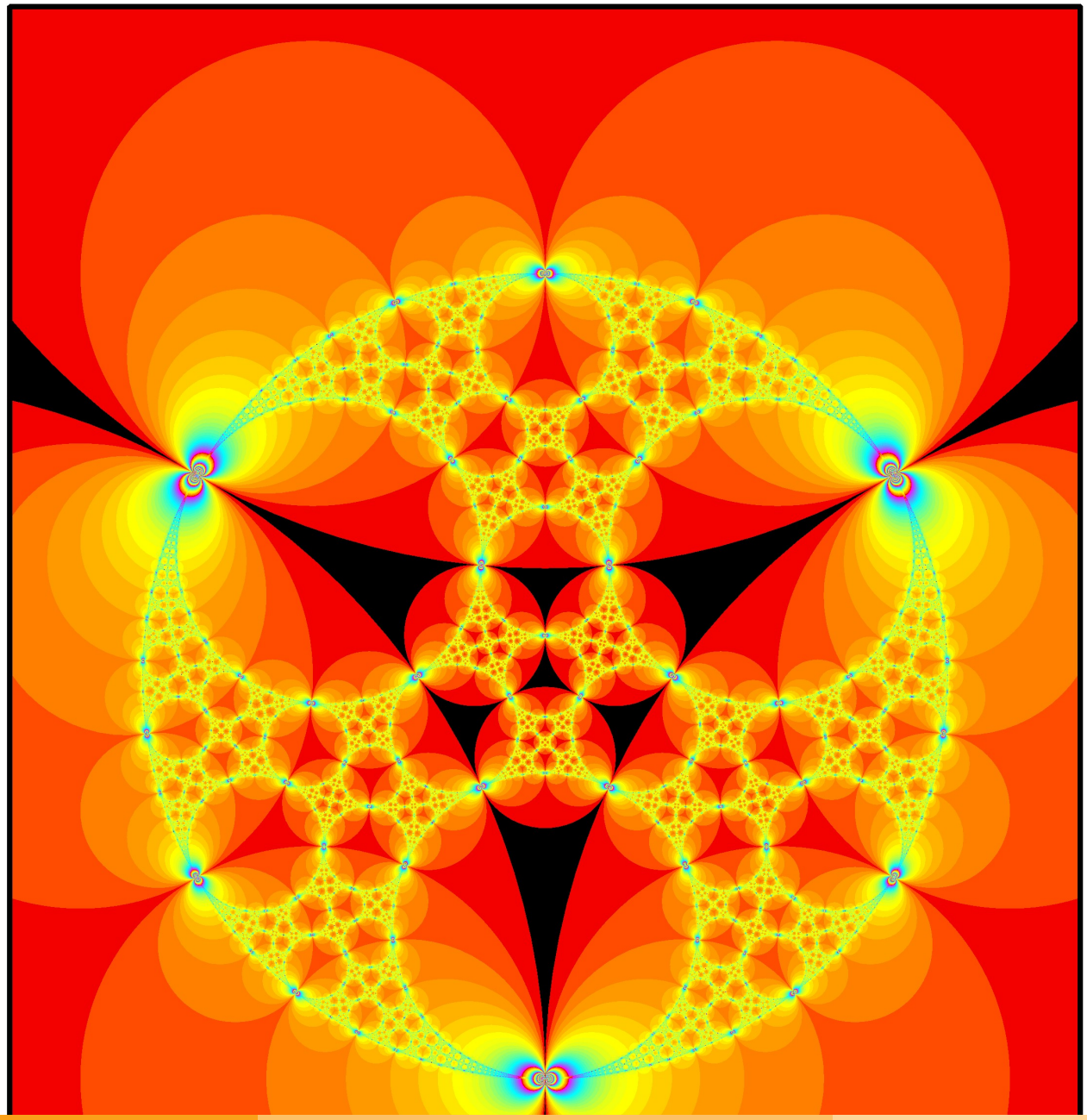
October 25, 2019

Apollonian Gasket

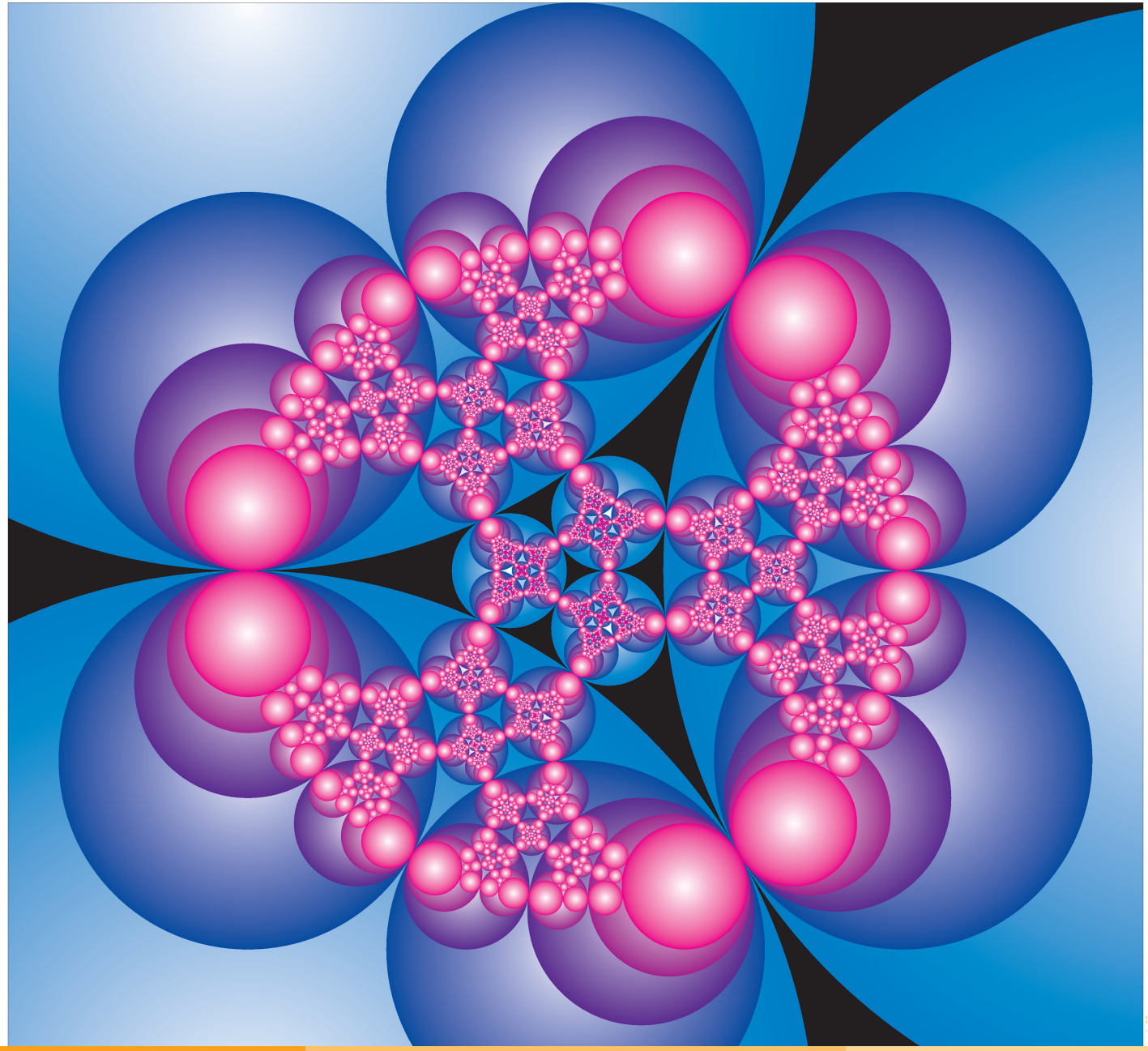


YouTube

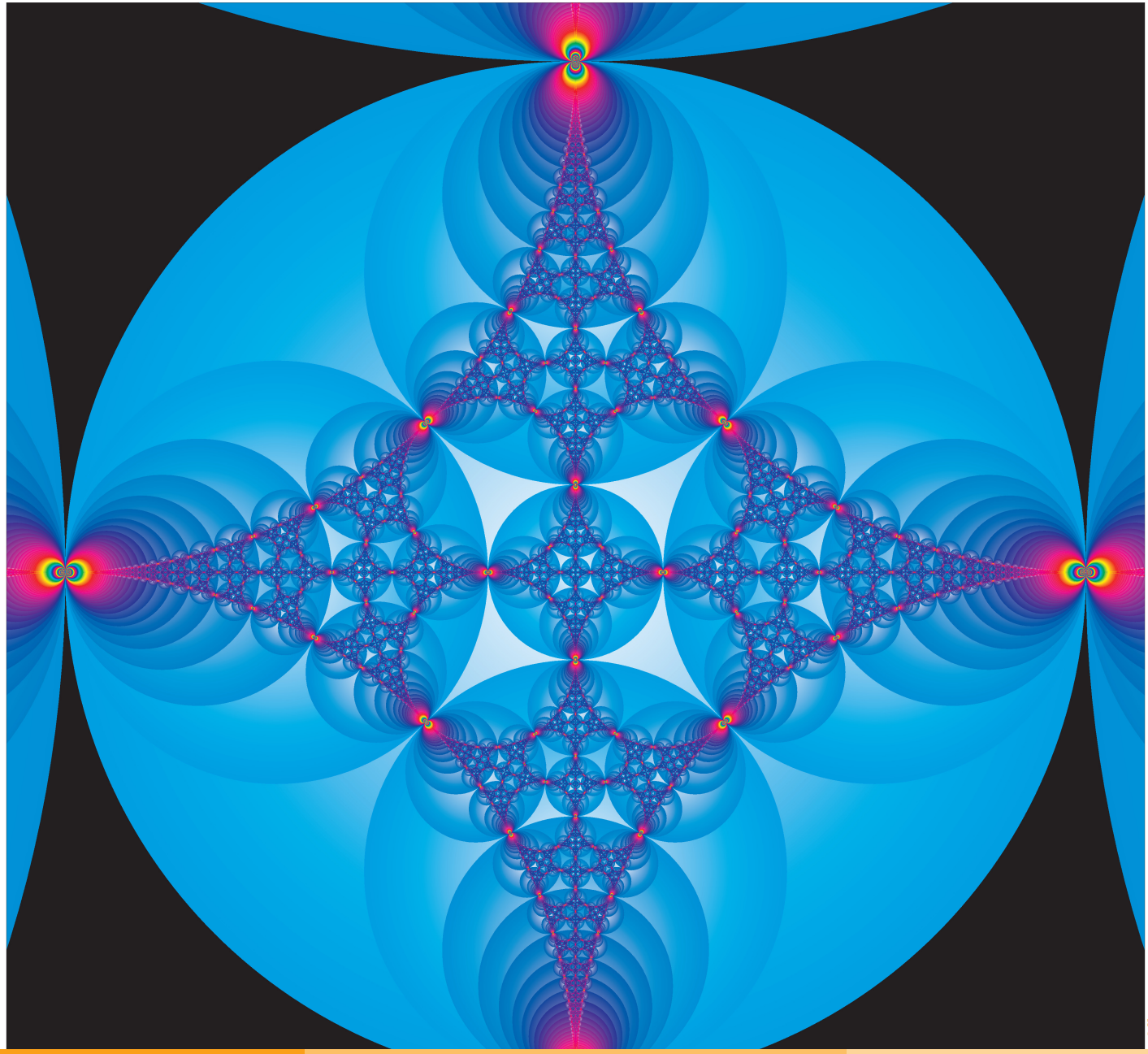
Glowing Gasket



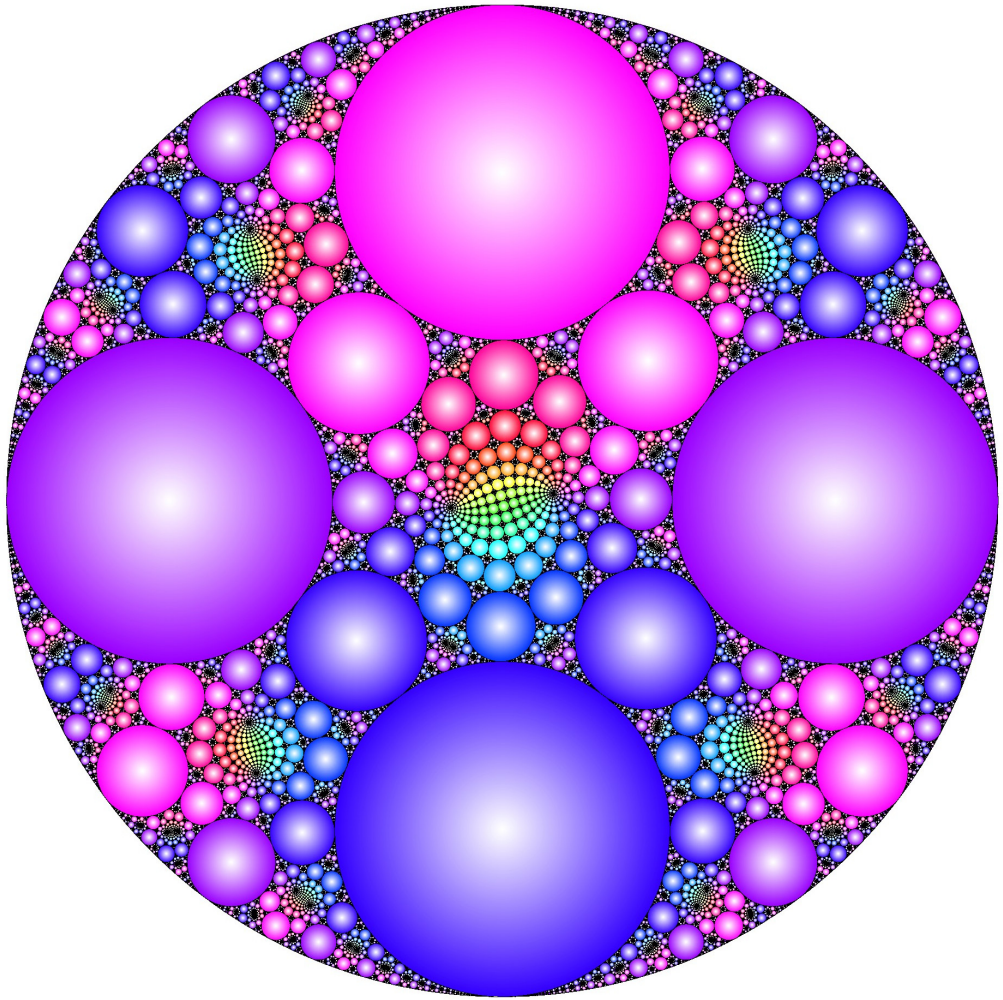
London Math Society Pearls



Dark Gasket

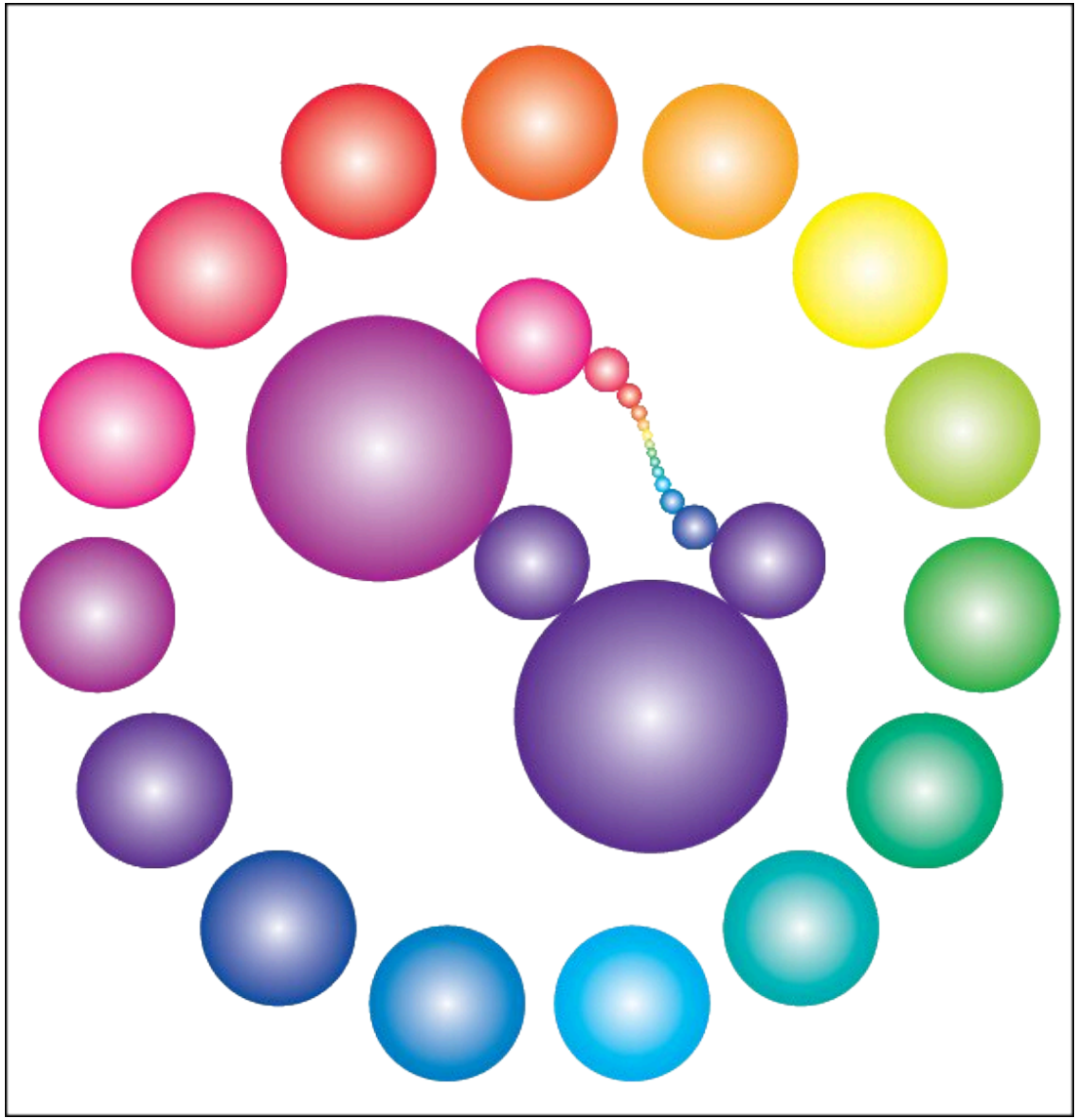


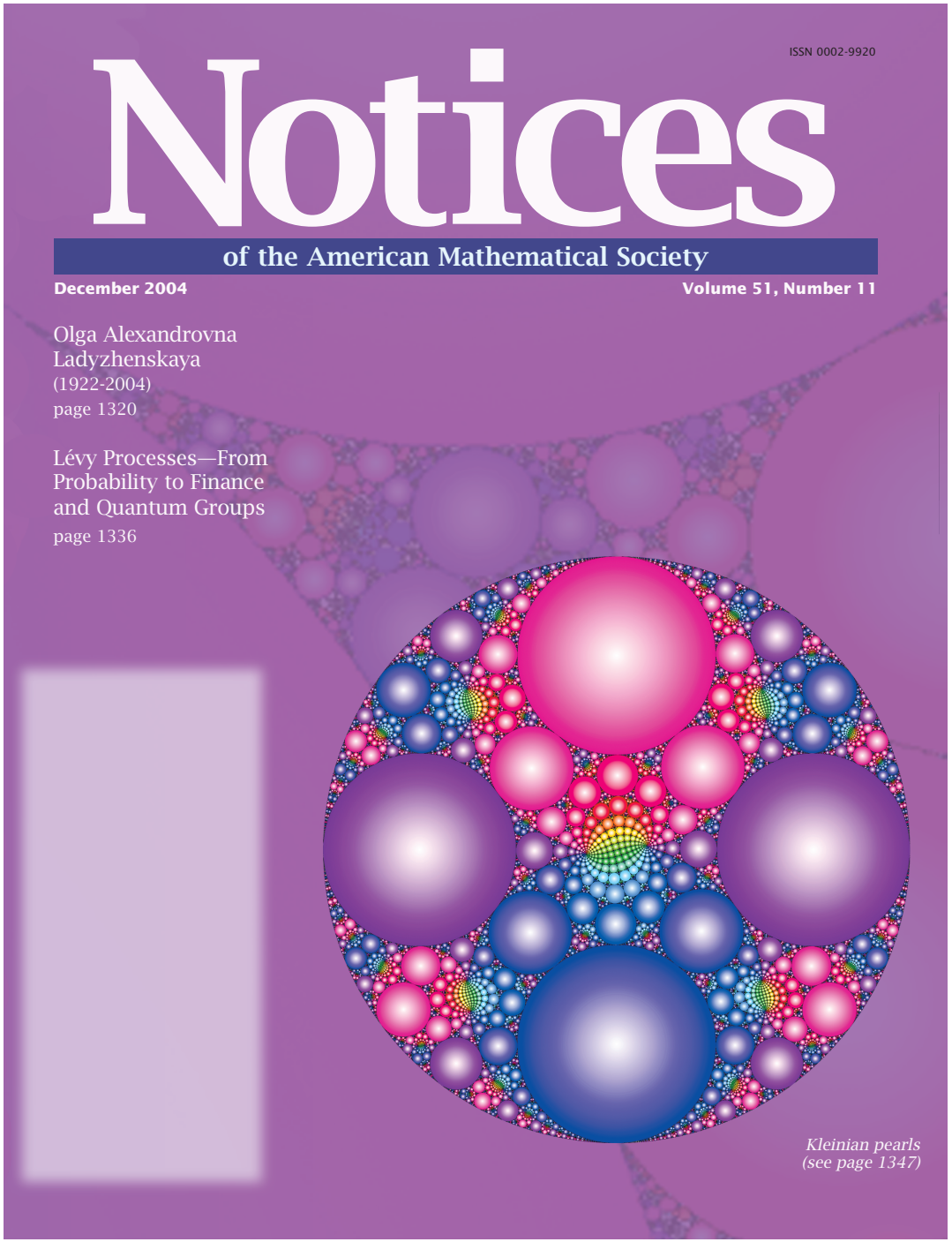
Maskit 1/15 Double Cusp



YouTube

Color Wheel

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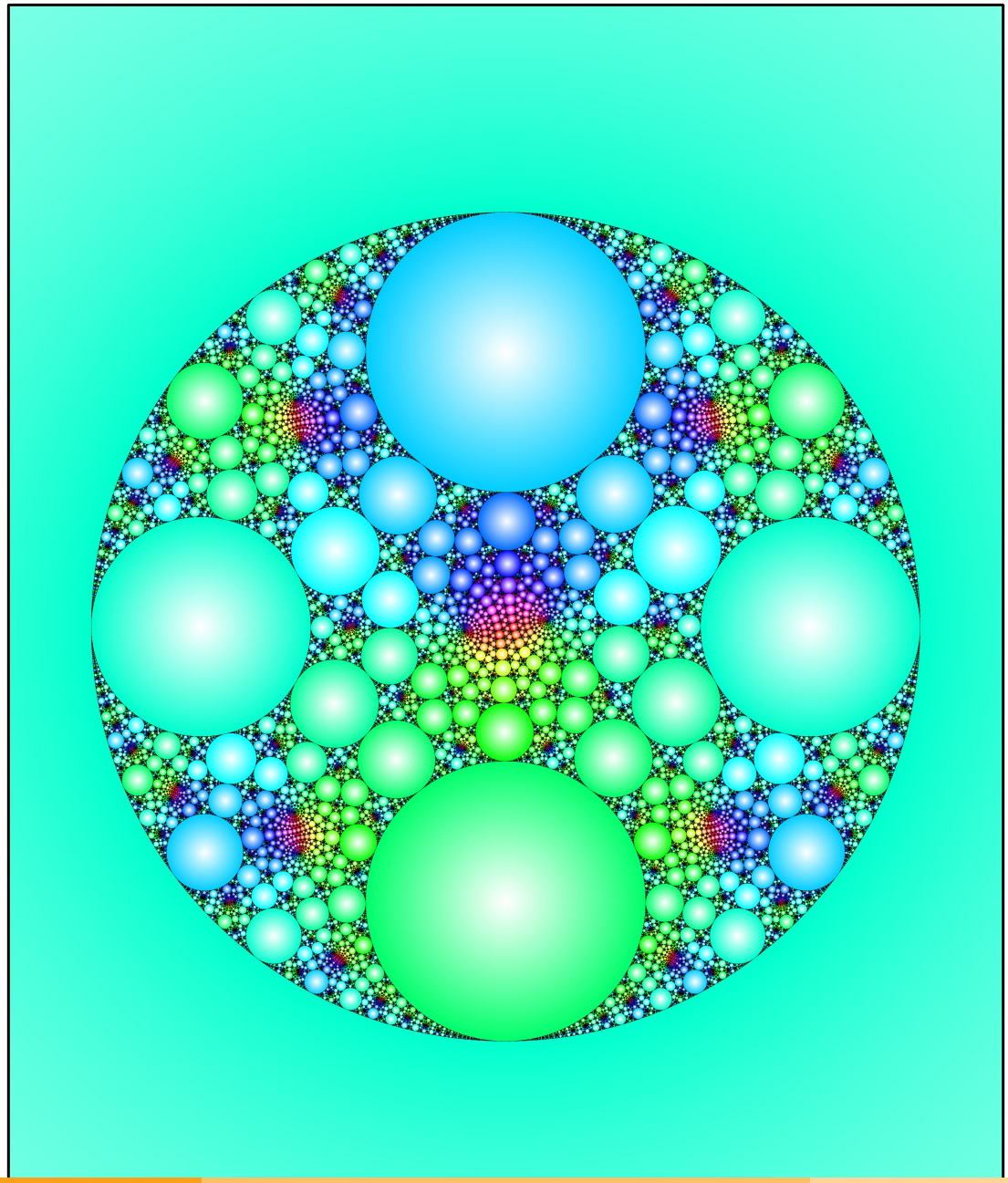
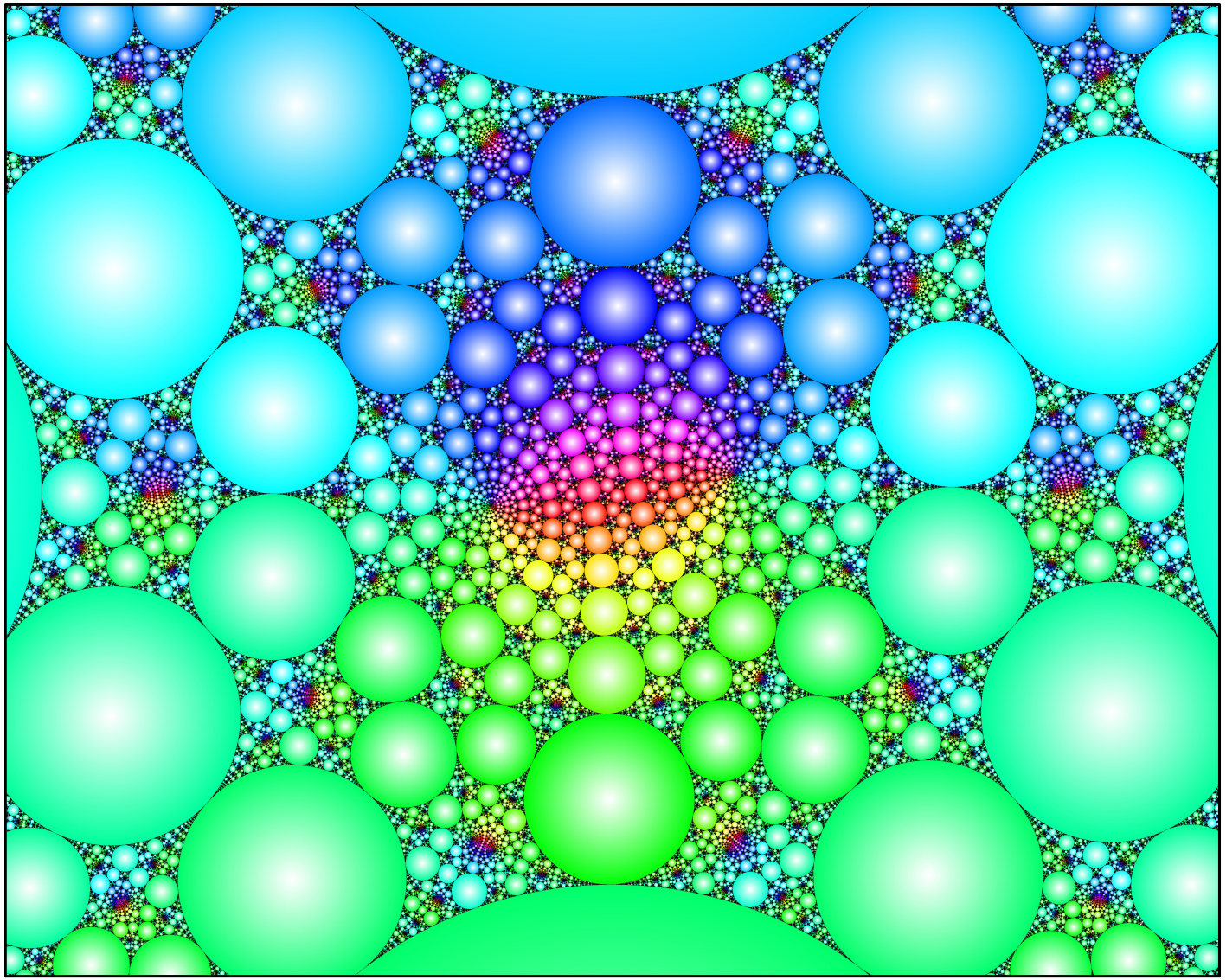
Notices
of the American Mathematical Society
December 2004
Volume 51, Number 11
Olga Alexandrovna Ladyzhenskaya (1922-2004) page 1320
Lévy Processes—From Probability to Finance and Quantum Groups page 1336
Kleinian pearls (see page 1347)

Maskit 2/31 Cusp Group





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$$m(H) = (m^{-1})^* H m^{-1}$$

Integrality

For the ring of integers O_K of an algebraic number field K, and a fractional O_K-ideal $\mathfrak{a}$, if an arrangement of disks in $\mathbb{P}^1(\mathbb{C})$ is generated by applying a subgroup of $\mathrm{PSL}_2(O_K)$ to a set of disks with forms $H \in M_{2,2}(\mathfrak{a})$, then

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- All the $\mathbb{C}$-**corners** w are in $\mathfrak{a}$.

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Proof: Distributive law of matrix multiplication.

Quadratic relations

$b(u, v) =$ real symmetric bilinear form in $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$.

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Just verify it for $m = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. (RREF)

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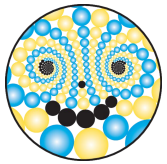

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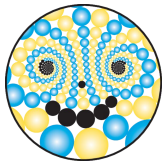

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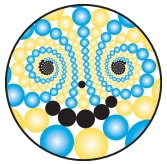

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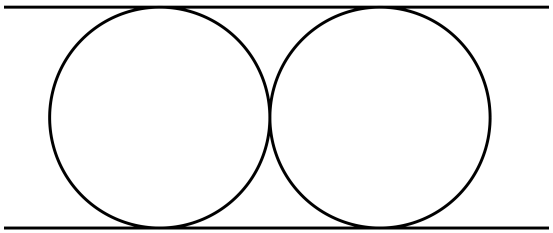

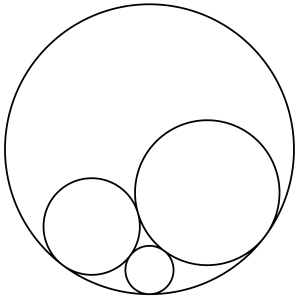

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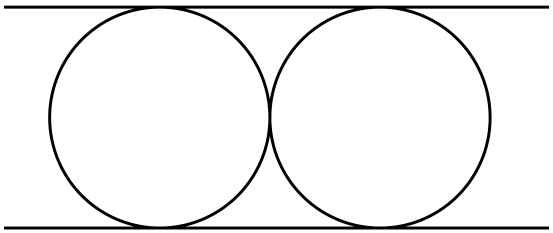

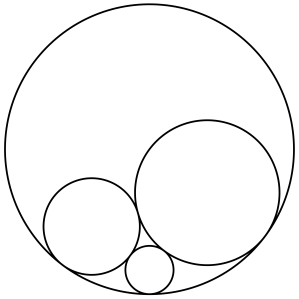

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$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 2 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 2 \end{bmatrix}, \begin{bmatrix} 2 & -2-i \\ -2+i & 2 \end{bmatrix}.$$

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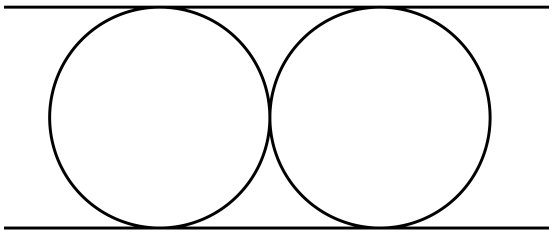

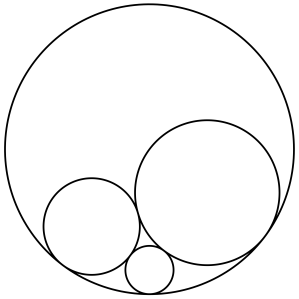

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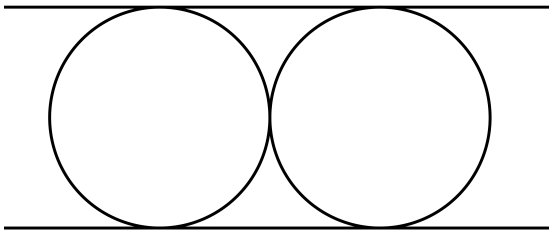

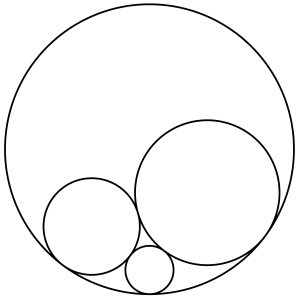

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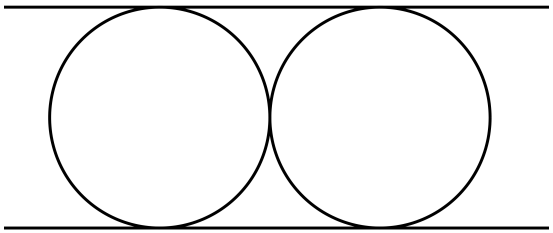

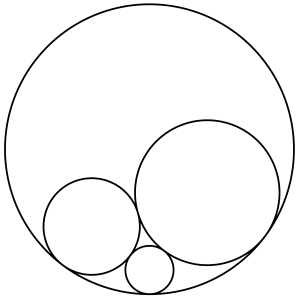

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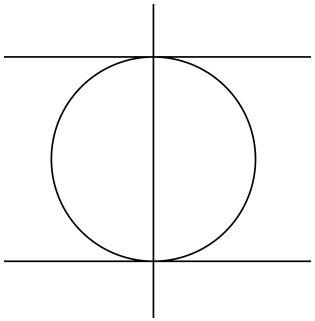
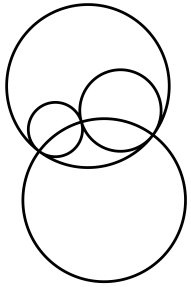
$2(s_1^2 + s_2^2 + s_3^2 + s_4^2) = (s_1 + s_2 + s_3 + s_4)^2$. (original version)

Hermitian Coxeter





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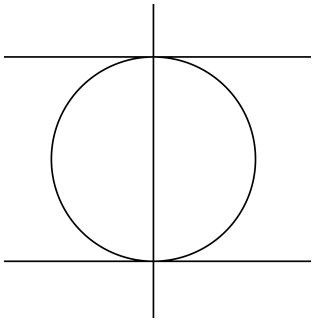

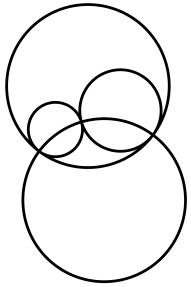

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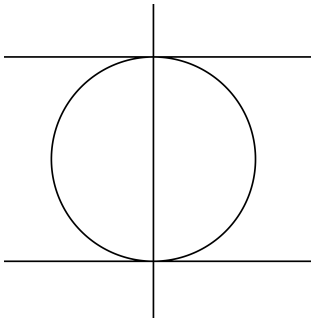

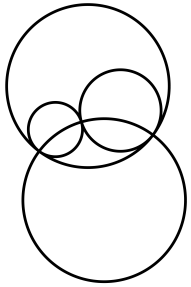

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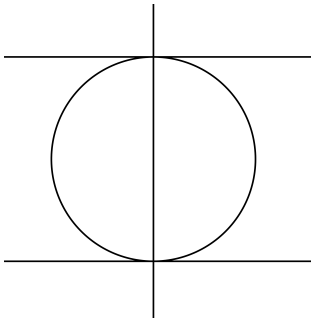

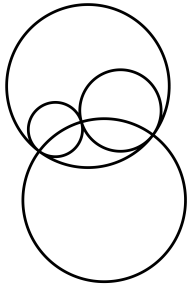

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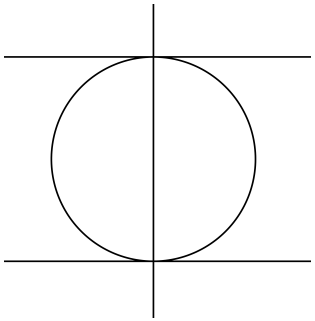
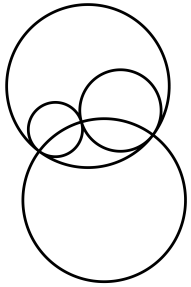
Hermitian Coxeter





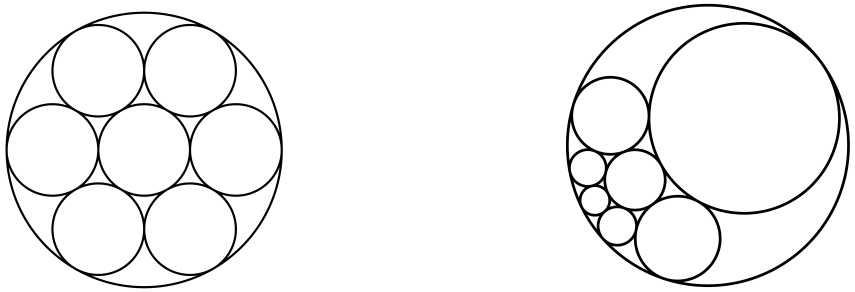
Standard Coxeter Forms: $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 2 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

$$b(u,v) = 2u_4v_4 - \sum_{1\le j<k\le 3} u_jv_k + u_kv_j.$$

$$2\overline{H_4}H_4 = \sum_{1\le j<k\le 3} \overline{H_j}H_k + \overline{H_k}H_j.$$

$s_4^2 = s_1s_2 + s_2s_3 + s_3s_1$. (original version)

Hermitian Steiner



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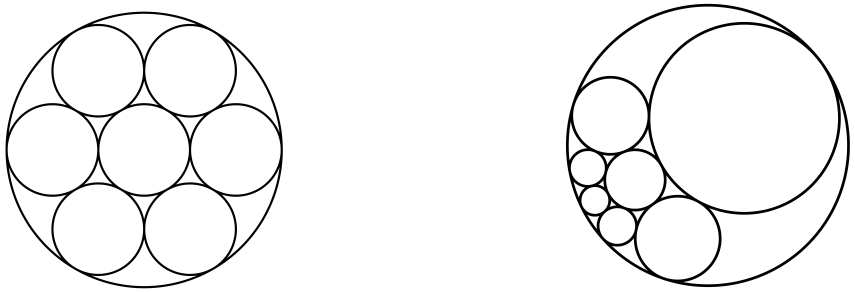

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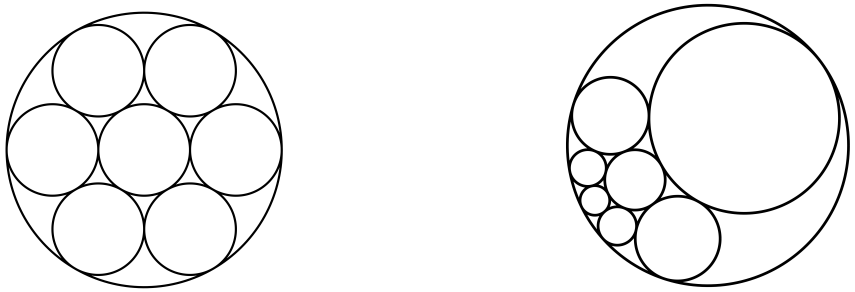

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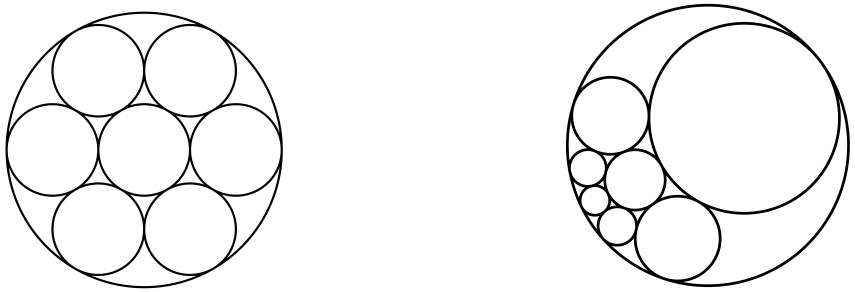

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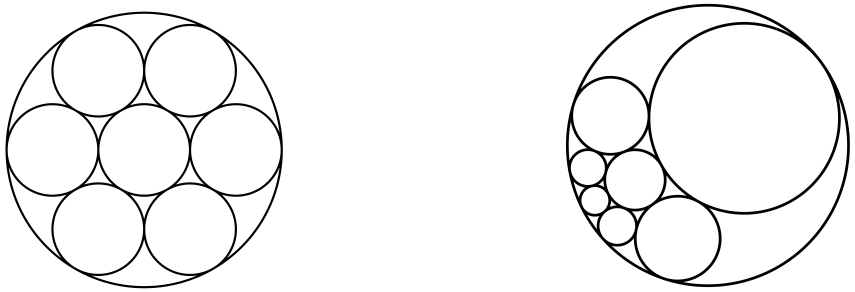

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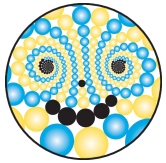

??

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- There are $3g - 3$ classes of words which are parabolic in G, up to conjugacy, inverse, and powers.
- Any two representations of G for which the same words in $a_j^{\pm 1}$ are parabolic are conjugate in $\mathrm{PSL}_2(\mathbb{C})$.

Limit set partitions and the circle web

- Floyd, 1980: continuous map $\phi : \overline{G} \to$ Limit Set $\Lambda(G)$, where $\overline{G} =$ **group completion** of G. (**infinite words** in G)

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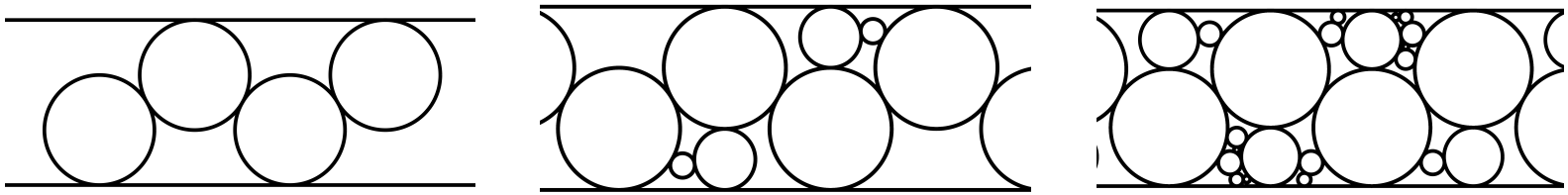
- The **junction circles** pass through three such fixed points, and are determined by those points. Each equivalence class of disks contains at least one junction disk.
- There is an algebraic number field K such that all the fixed points of parabolic elements may be chosen in K. ("field of definition")

Maskit 1/2 Cusp



- Gens: $a(z) = 1 + \sqrt{-3} + \dfrac{1}{z}$, $\quad b(z) = z + 2$. Parabolic: Ba^2, $BAba$, b.

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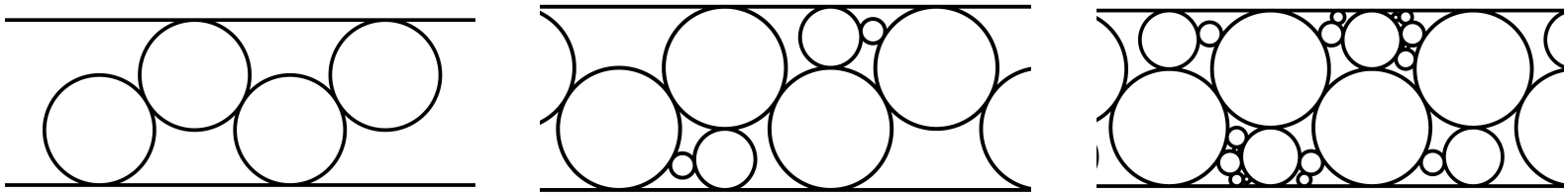

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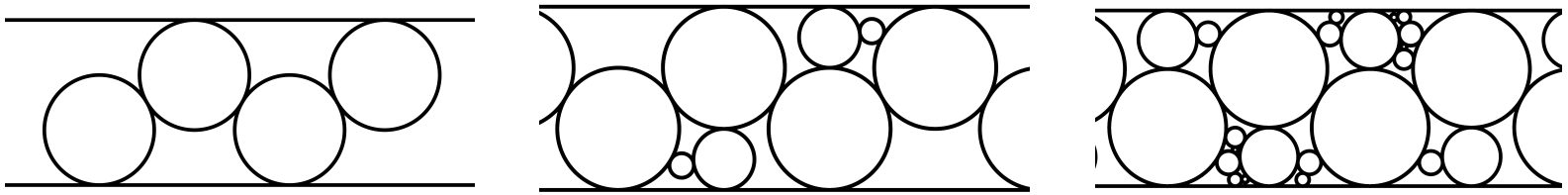

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- Curvatures: $n\sqrt{3}$, $n = 0,1,2,5,6,8,9,12,13,17,18,20,21,22,24,25,29,\ldots$

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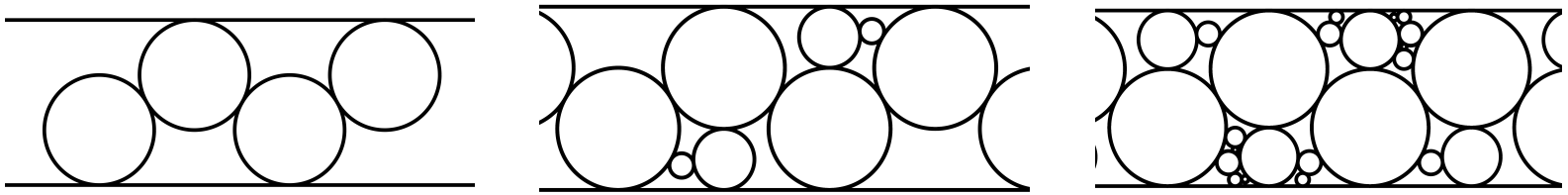

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- $\mathbb{C}$-corners: all of form $m\sqrt{3} + ni$, $m, n \in \mathbb{Z}$.

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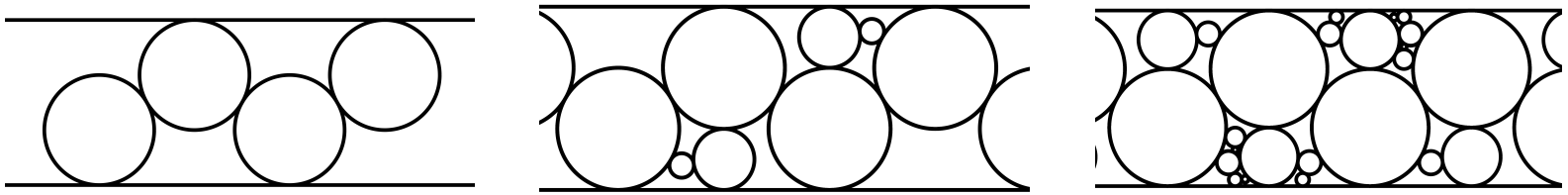

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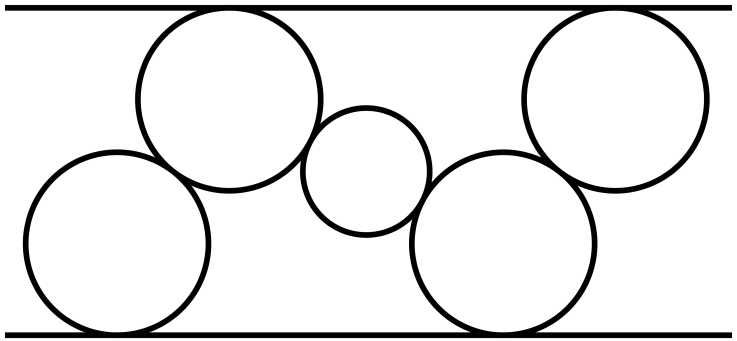
- Recursion: $C_0 + C_2 = 2L_0 + 2C_1$, $C_1 + C_3 = 2L_1 + 2C_2$.

Maskit 1/3 Cusp



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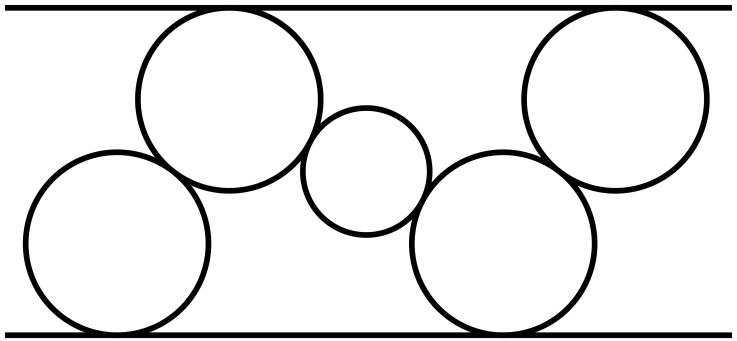

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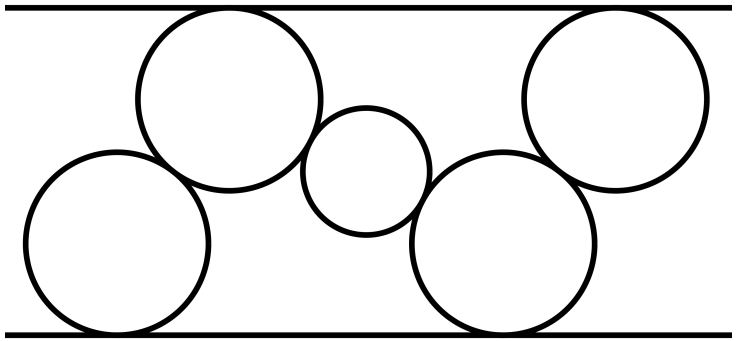

- Parabolic: Ba^3, $BAba$, b.
- Gens: $b(z) = z + 2$,

$$a(z) = \mu + \frac{1}{z} = \frac{4 - \sqrt{2\sqrt{41} - 10} + i\left(2 + \sqrt{10 + 2\sqrt{41}}\right)}{4} + \frac{1}{z}.$$

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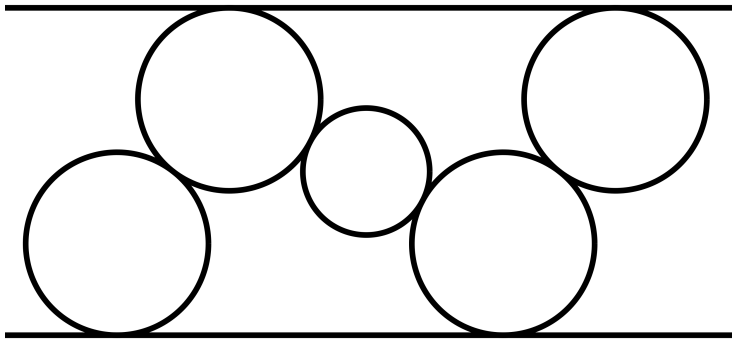

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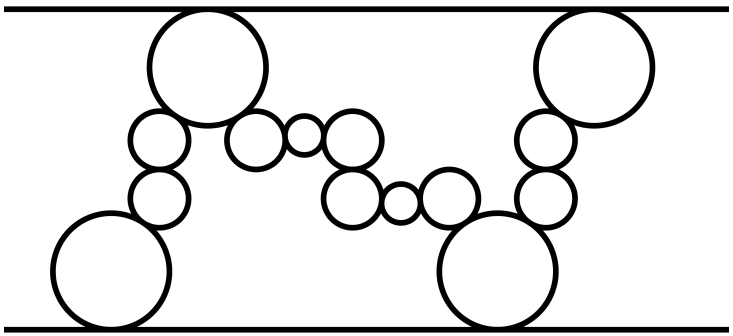
- $\mu^2 - (2 + i)\mu + 2 + 2i = 0$.

Maskit $3/10 = \frac{1}{3+\frac{1}{3}}$ Cusp



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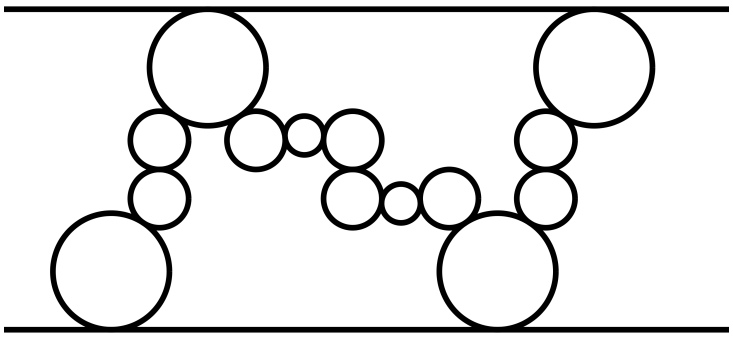

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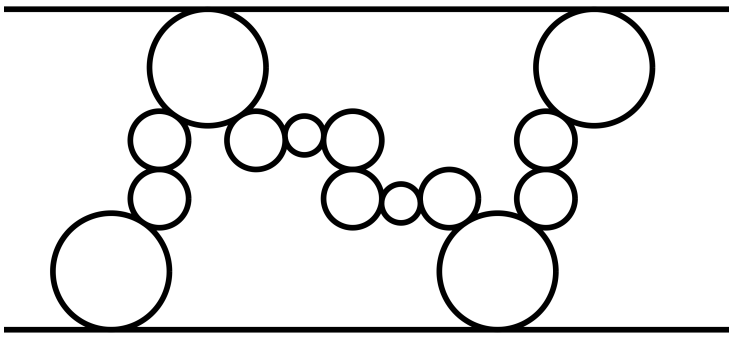

- Parabolic: $Ba^4Ba^3Ba^3$, $BAba$, b.
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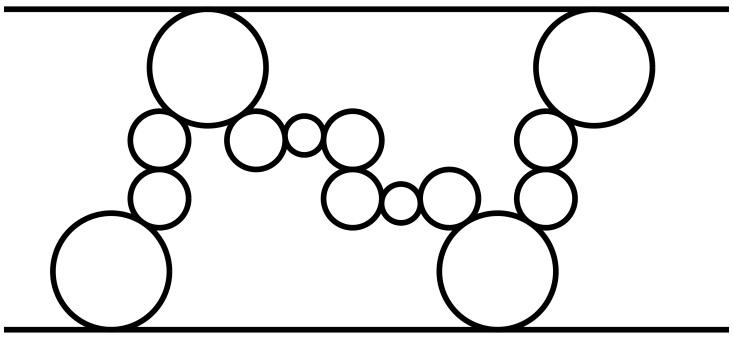

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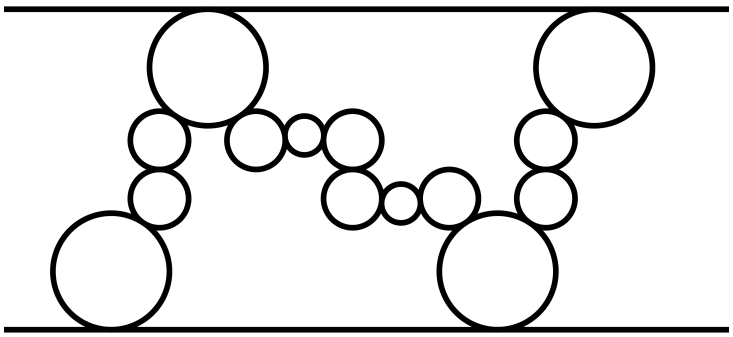

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- Circles $L_0 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $L_1 = \begin{bmatrix} 0 & -i \\ i & \sqrt{11} \end{bmatrix}$, $C_0 = \begin{bmatrix} \sqrt{11} & -i+\sqrt{11} \\ i+\sqrt{11} & \sqrt{11} \end{bmatrix}$, ...

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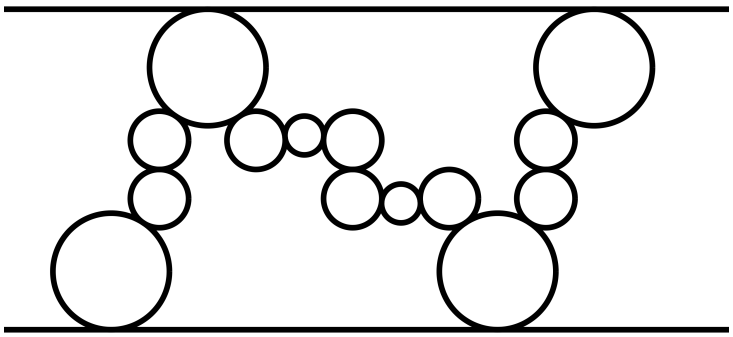

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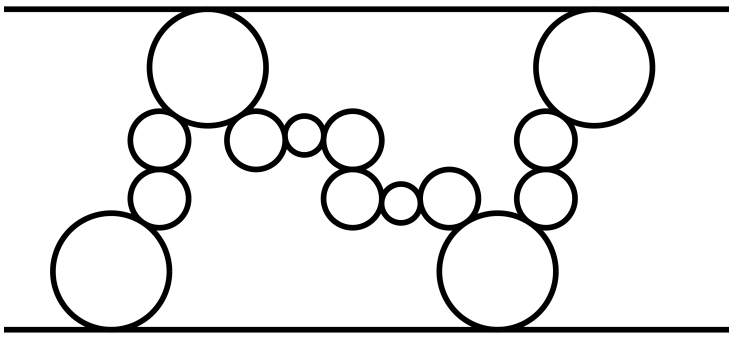
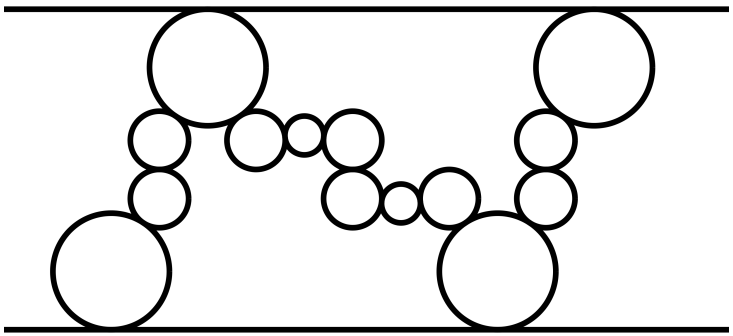
- $\mu^2 - \mu + 3 = 0$.
- Circles $L_0 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $L_1 = \begin{bmatrix} 0 & -i \\ i & \sqrt{11} \end{bmatrix}$, $C_0 = \begin{bmatrix} \sqrt{11} & -i+\sqrt{11} \\ i+\sqrt{11} & \sqrt{11} \end{bmatrix}$, ...
- Curvatures: $n\sqrt{11}$, $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, \ldots$

Maskit $3/10 = \frac{1}{3+\frac{1}{3}}$ Cusp



- Parabolic: $Ba^4Ba^3Ba^3$, $BAba$, b.
- Gens: $b(z) = z + 2$, $a(z) = \mu + \dfrac{1}{z} = \dfrac{1+\sqrt{-11}}{2} + \dfrac{1}{z}$.
- $\mu^2 - \mu + 3 = 0$.
- Circles $L_0 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $L_1 = \begin{bmatrix} 0 & -i \\ i & \sqrt{11} \end{bmatrix}$, $C_0 = \begin{bmatrix} \sqrt{11} & -i+\sqrt{11} \\ i+\sqrt{11} & \sqrt{11} \end{bmatrix}$, ...
- Curvatures: $n\sqrt{11}$, $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, \ldots$
- $\mathbb{C}$-corners: of form $\dfrac{m\sqrt{11}+ni}{2}$, $n \equiv m \pmod 2$.

Maskit $3/10 = \frac{1}{3+\frac{1}{3}}$ Cusp



- Parabolic: $Ba^4Ba^3Ba^3$, $BAba$, b.
- Gens: $b(z) = z + 2$, $a(z) = \mu + \dfrac{1}{z} = \dfrac{1+\sqrt{-11}}{2} + \dfrac{1}{z}$.
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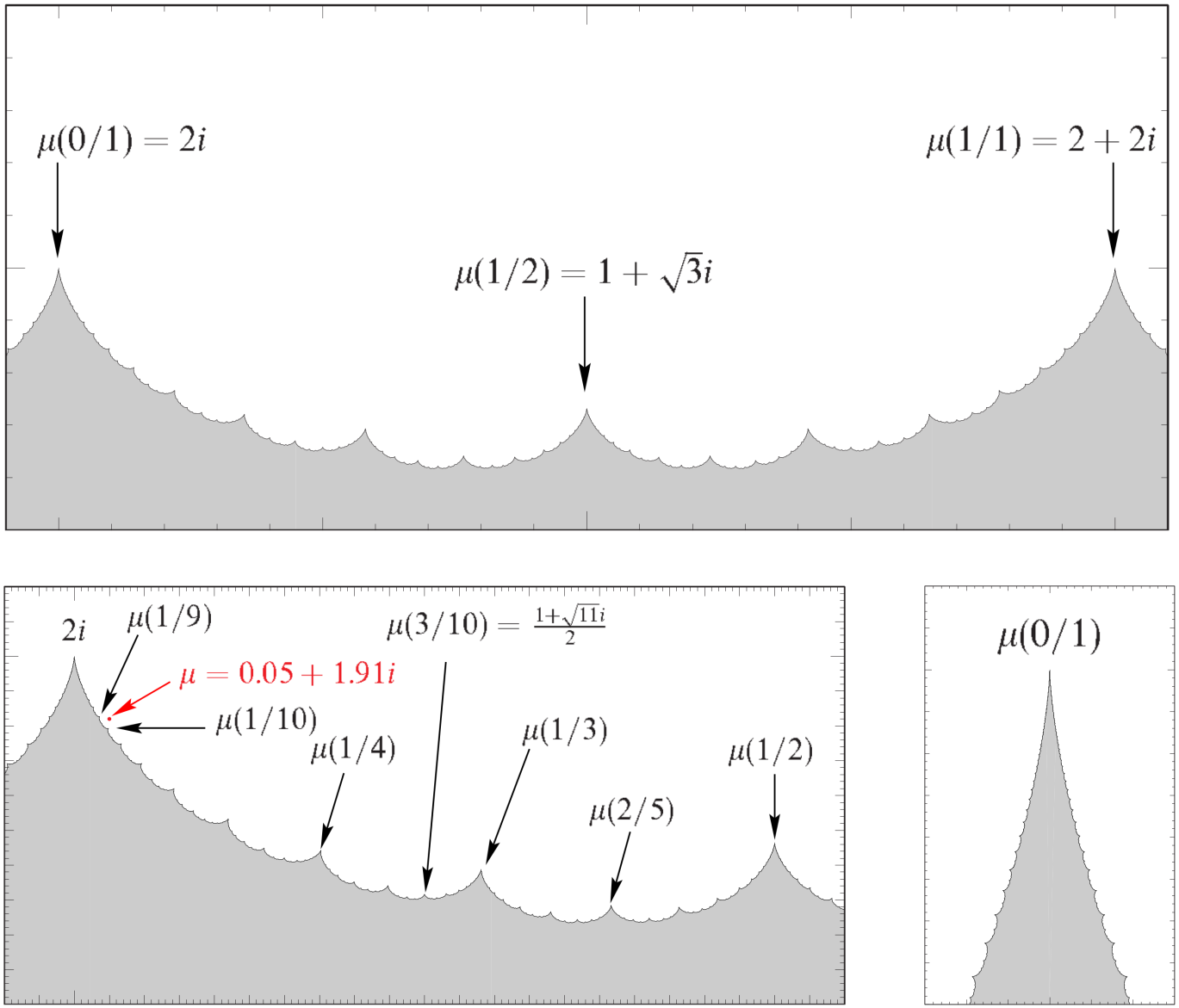
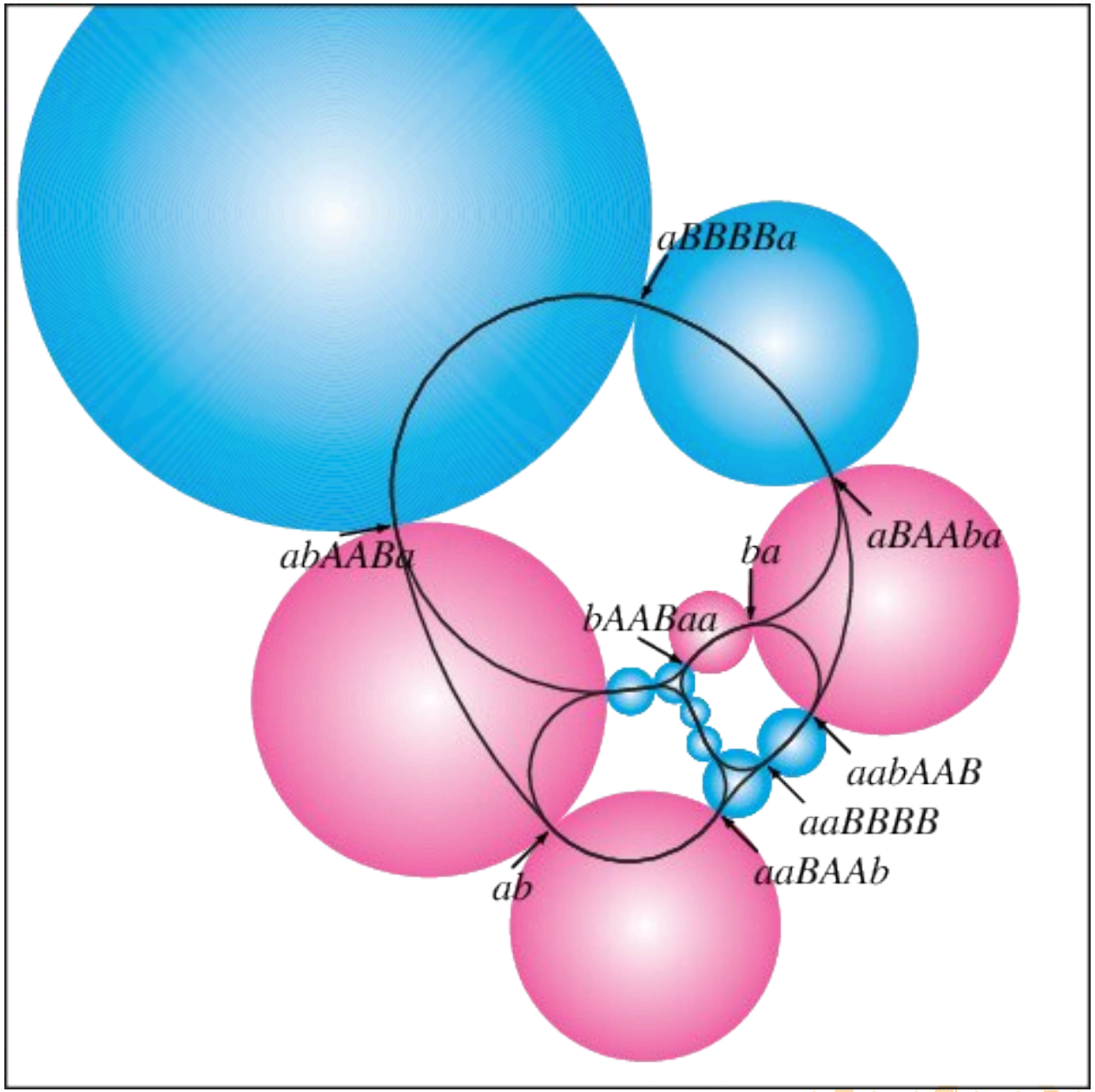
- Circles $L_0 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $L_1 = \begin{bmatrix} 0 & -i \\ i & \sqrt{11} \end{bmatrix}$, $C_0 = \begin{bmatrix} \sqrt{11} & -i+\sqrt{11} \\ i+\sqrt{11} & \sqrt{11} \end{bmatrix}$, ...
- Curvatures: $n\sqrt{11}$, $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, \ldots$
- $\mathbb{C}$-corners: of form $\dfrac{m\sqrt{11}+ni}{2}$, $n \equiv m \pmod 2$.
- Recursions: $4C_0 + C_4 = 3L_0 + 3C_1$, $C_0 + C_3 = L_1 + C_1 = L_0 + C_2, \ldots$

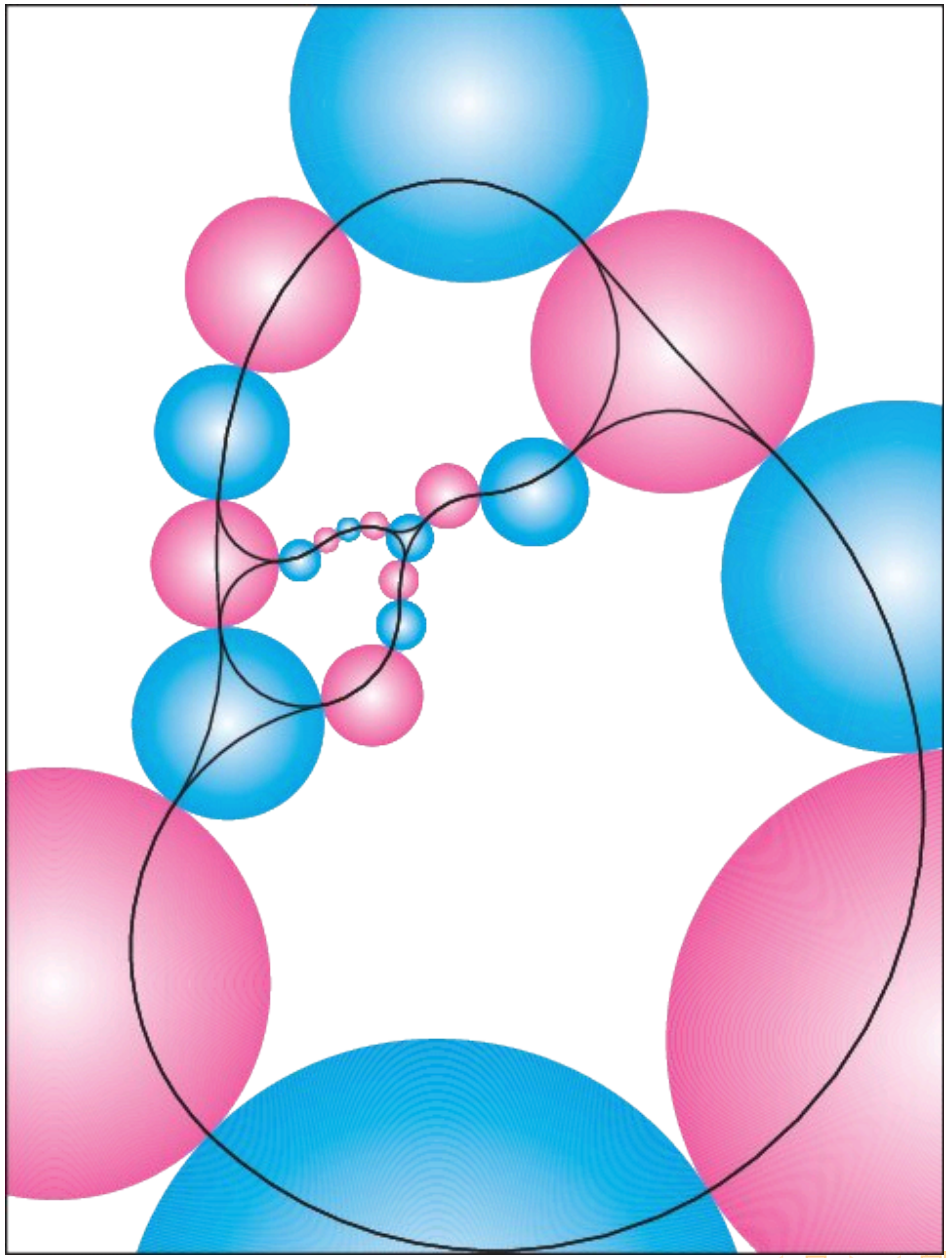
Map of Maskit Cusps



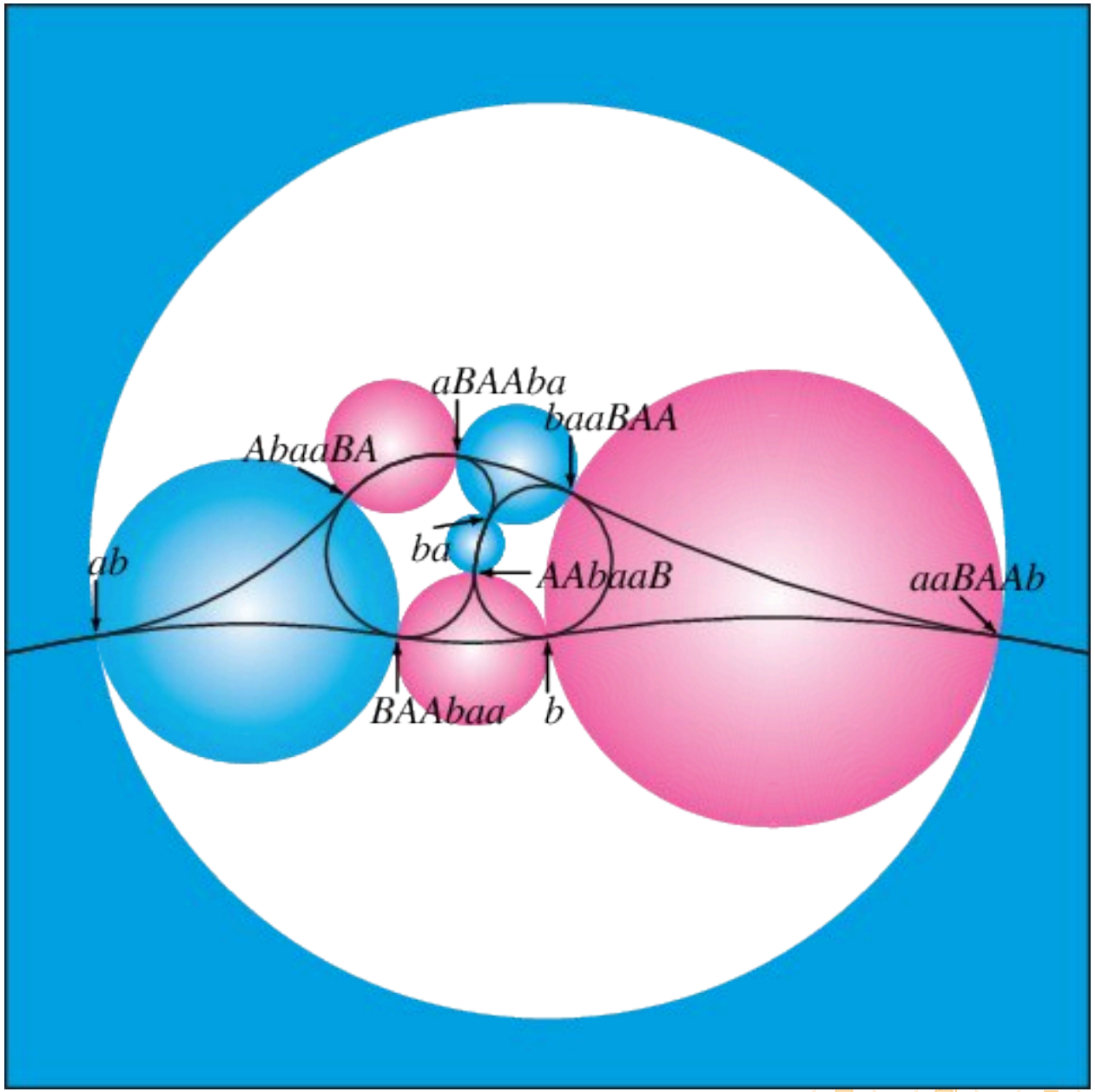
Genus 2 Maximal Cusp



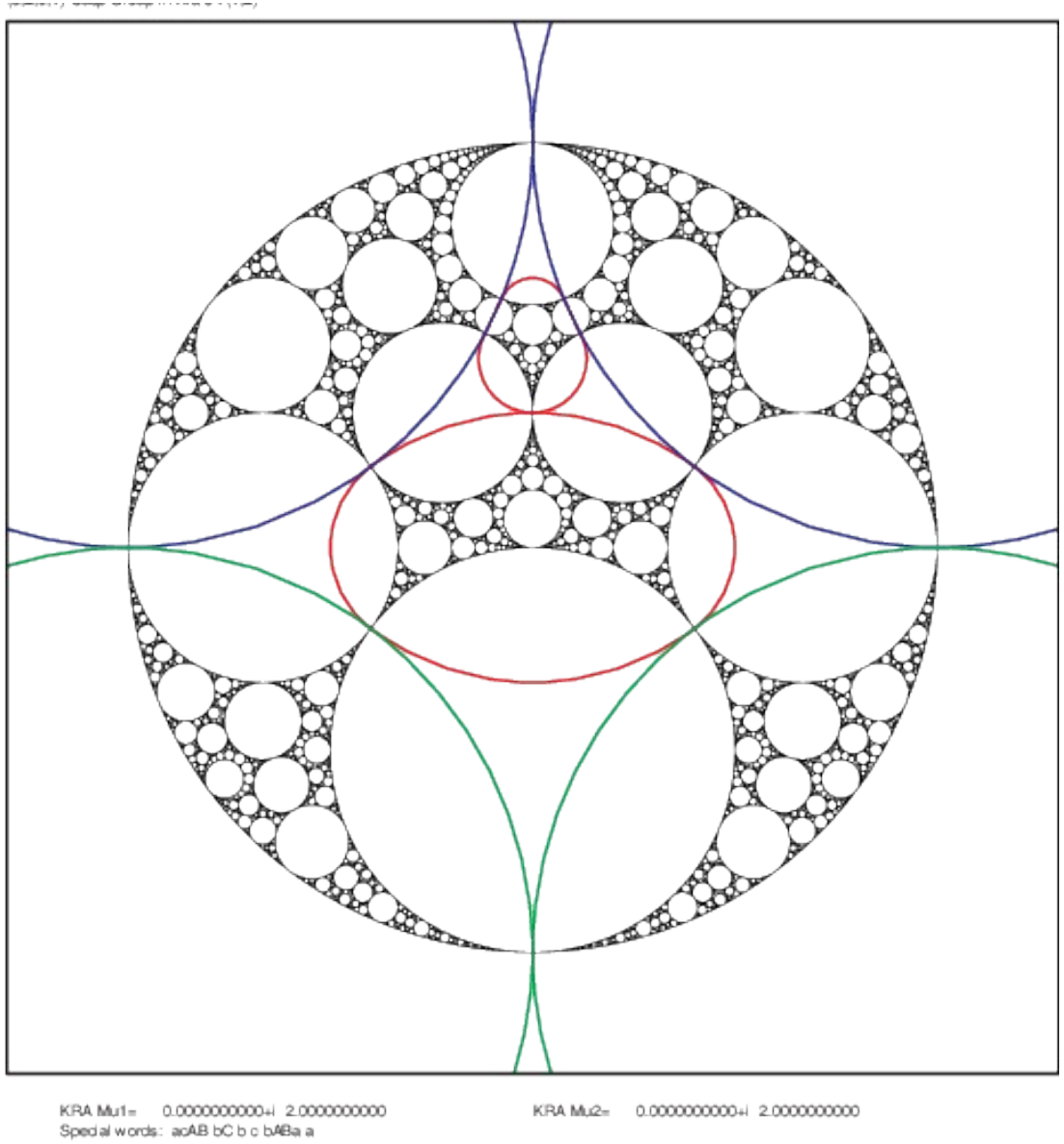
Genus 2 Maximal Cusp



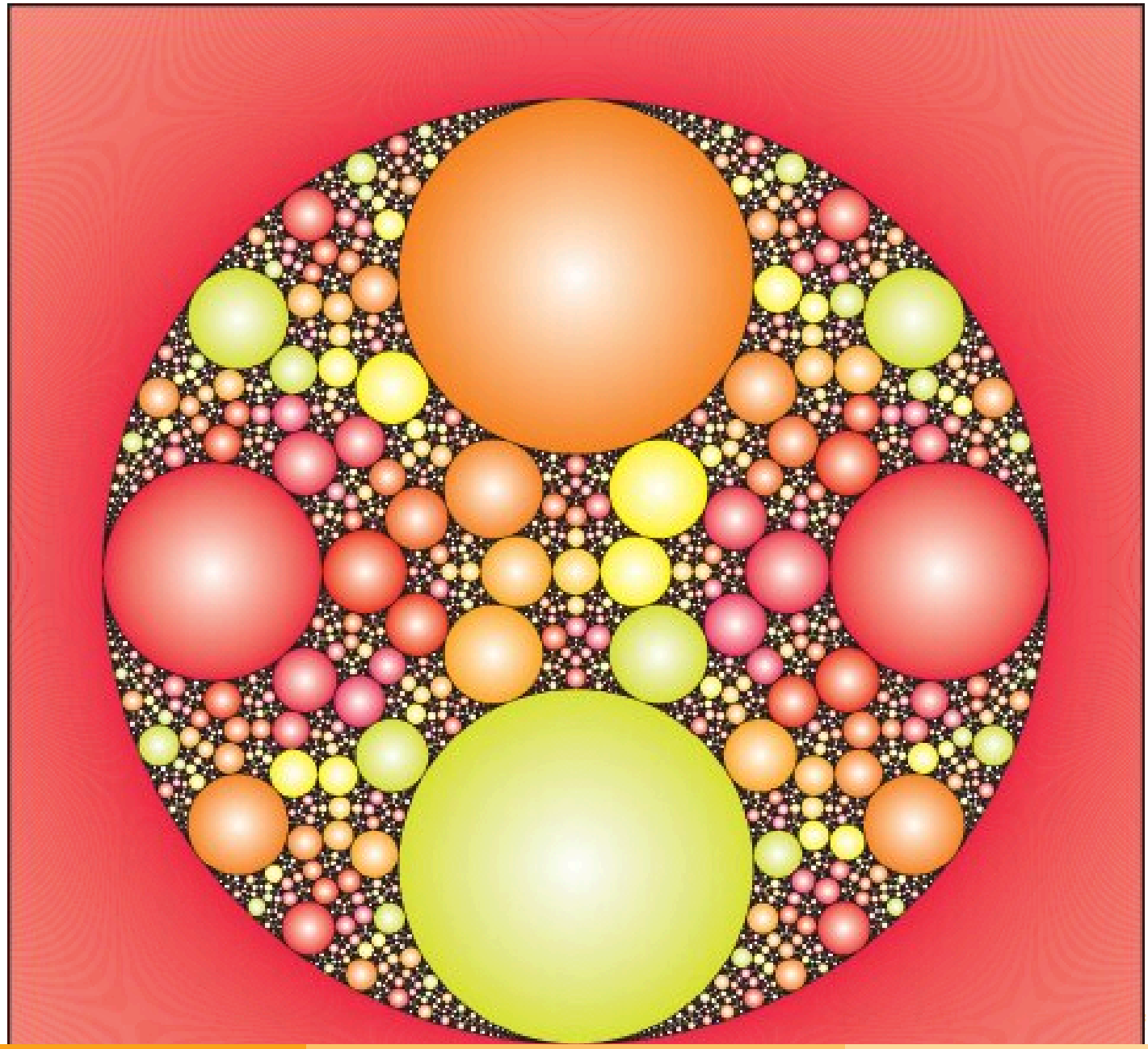
Genus 2 Maximal Cusp



Genus 3 Maximal Cusp



Maskit 3/10 Fruit Salad



Thanks!